PROBLEM solvers

This month marks the inception of a new Mathematics Teacher *department on problem solving modeled on a similar department in our sister journal* Teaching Children Mathematics. *Problem Solvers will appear in the August and April issues of* MT *and will present interesting problems to share with students. Submit your students' work, and creative solutions will be published in the journal or posted on the* MT *website. To initiate the department, its editors have presented a problem and their own students' solutions this month, together with a prompt for readers to try with their students. Solutions must reach the department editors by December 1, 2011, to be considered for publication.*

Edited by ***Seán P. Madden***
smadden1@greeleyschools.org
Greeley West High School
Greeley, CO

Ricardo Diaz
Ricardo.diaz@unco.edu
School of Mathematical Sciences
University of Northern Colorado
Greeley, CO

THE 5-HORSE RACE
The Problem
In a 5-horse race, how many different finishing orders are there?

A False Start

Students with a typical background in counting might rush to the incorrect solution of ${}_5P_5 = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ distinct permutations. At this point, students should be prompted to think about why this answer is not correct. Once they realize that horses can finish in a tie, they are ready to work on a solution. In the classroom, one way to approach this difficult counting problem is to make it a simpler problem by reducing the number of horses in the race. Seeing how students approach the problem for 2 and 3 horses might suggest strategies for the cases of 4 or 5 horses. We assume that this discussion has already taken place, as we look at the 5-horse problem.

Back to the Starting Gate

Starting fresh, let's think about some simple examples of ties. One possibility is a five-way dead heat. A five-way tie can occur in only one way. Of course, there are other ways to tie, and at this point the problem begins to invite real mathematical creativity, the kind of creativity that we hope your students will allow us to showcase in future columns.

A SAMPLING OF STUDENT SOLUTIONS

Solution 1

The first solution comes from a pair of second-year algebra students, Michal and Aleck. They drew a set of diagrams for a variety of cases and then applied counting principles to add the various cases. After checking their work several times, they arrived at a total number of 541 finishes for a 5-horse race. **Figure 1** shows an excerpt of their work.

Michal and Aleck were so captivated by this problem that they were willing to spend hours of persistent work drawing the diagrams, analyzing the patterns, and extrapolating from these patterns to arrive at a number for each set of cases. They arrived at a correct solution without creating a recurrence relation or relying on formulas. Had the problem been posed for a 6-horse race, however, it seems likely that they would have found the problem intractable.

Solution 2

One of the department editors also chose to draw diagrams for each case but, rather than count each case, applied combinations and permutations to arrive at a final answer. In this analysis, the problem is broken into cases according to which kinds of ties can happen (see **fig. 2**).

Case 1, shown in column 1, corresponds to the situation in which each of the 5 horses crosses individually.

Case 2 corresponds to the situation in which all 5 horses cross the finish line simultaneously. There is only one way this can happen.

Case 3 considers the possibility of 4 horses finishing simultaneously and 1 horse finishing singly. Any one of the 5 horses might be the singleton, and this horse might finish behind the others or, as Secretariat did, ahead of the pack by thirty-one lengths. We may thus conclude that case 3 includes 10 finishes.

Cases 4(a) and 4(b) consider 3 horses finishing simultaneously. In case 4(a), the remaining horses finish singly. These three sets of horses may then be permuted, so $({}_5C_3)({}_3P_3) = 60$ unique finishes. In case 4(b), the remaining 2 horses also tie (in either first or second place). Computing ${}_5C_3 = 10$ gives the number of ways of selecting 3 horses in a tie. If the other 2 horses also tie, we may simply multiply by 2 to obtain another 20 possibilities.

Case 5(a) can be solved in a manner similar to that for case 4(a). First, we select the 3 horses that finish singly. The remaining 2 horses tie in first, second, third, or fourth place. Thus, $({}_5C_2)({}_4P_4) = 240$ unique finishes.

Case 5(b) seems to present the real challenge and deserves special consideration. In case 5(b), there are two sets of two-way ties. We cannot simply select 2 horses out of 5, then select 2 horses from the 3 that remain, and then multiply by ${}_3P_3$ to take into account the singleton. Doing so results in an overcount. An alternative is to list all combinations of 2 horses with one singleton, eliminate the degenerate cases, and then multiply the unique sets of pairs of two-way ties by ${}_3P_3$. The result yields 90 unique finishes and is summarized in **figure 3**. To account for the degenerate possibilities, we now give each horse a letter designation from F to J and eliminate the degenerate cases by striking them through. We then sum the solutions for all cases (see **fig. 4**).

Solution 3

A similar approach, developed by the second department editor, is shown in **figure 5**. Each possible outcome of the race is described by a configuration of squares. In each configuration, the

Fig. 1 Excerpts of the work done by two second-year algebra students seem to indicate that using this method with 6 horses would have created a prohibitive amount of work.

Case 1	Case 2					Case 3				Case 4(a)			Case 4(b)			Case 5(a)		Case 5(b)	
X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
X						X				X			X	X		X		X	X
X										X						X		X	
X																X			
X																			

Fig. 2 Each **X** represents a horse. As each case is considered, the distinguishability of these horses is taken into account.

FG HI J	~~FG~~ ~~J~~ ~~HI~~	FG I HJ	~~FG~~ ~~HJ~~ ~~I~~	FG H IJ	~~FG~~ ~~IJ~~ ~~H~~
FH J GI	FH I JG	FH G IJ			
FI J HG	FI H JG	FI G HJ			
FJ G HI	FJ H GI	FJ I GH			
GH F IJ	~~GH~~ ~~I~~ ~~FJ~~	~~GH~~ ~~J~~ ~~FI~~			
GI F HJ	~~GI~~ ~~H~~ ~~FG~~	~~GI~~ ~~J~~ ~~FH~~			
GJ F HI	~~GJ~~ ~~H~~ ~~FI~~	~~GJ~~ ~~I~~ ~~HF~~			
~~HJ~~ ~~F~~ ~~GI~~	~~HJ~~ ~~G~~ ~~FI~~	~~HJ~~ ~~I~~ ~~FG~~			

Fig. 3 Each set that is not eliminated is multiplied by ${}_3P_3 = 6$ to give a total of 90 finishes.

horizontal axis represents time, and the vertical stacks represent ties. (This approach reverses the conventions used in the second solution, in which the vertical axis is time.) The different configurations have been sorted into color-coded categories. Each configuration has a caption that describes the number of distinct ways that horse names can

	Number of Unique Finishes
Case 1	120
Case 2	1
Case 3	10
Case 4(a)	60
Case 4(b)	20
Case 5(a)	240
Case 5(b)	90
Total	**541**

Fig. 4 All possible unique finishes are enumerated.

be assigned to that configuration. This number is found by first considering all possible 5! permutations of the 5 horse names and then converting each stack of *ordered* names that occur in a tie into an *unordered* list by dividing by the appropriate factorial. (These quotients of factorials appearing in the captions are often called *generalized binomial coefficients* or *multinomial coefficients*.)

Solution 4

The fourth solution depends on a recurrence relation between the solution to the 5-horse problem and solutions to versions of the problem that involve fewer horses. Letting H_n represent the number of ways in which horses can finish in an n-horse race and $C(n, k)$ represent the binomial coefficients, we have the following:

$$H_5 = C(5, 5) + C(5, 4) \cdot H_1 + C(5, 3) \cdot H_2 + C(5, 2) \cdot H_3 + C(5, 1) \cdot H_4$$
$$H_4 = C(4, 4) + C(4, 3) \cdot H_1 + C(4, 2) \cdot H_2 + C(4, 1) \cdot H_3$$
$$H_3 = C(3, 3) + C(3, 2) \cdot H_1 + C(3, 1) \cdot H_2$$
$$H_2 = C(2, 2) + C(2, 1) \cdot H_1$$

Checking that $H_1 = 1$ is easy, so we can unravel this chain of equations to solve for all the values up through $H_5 = 541$. This process can be repeated to determine higher values of H_n as well.

Here is an explanation of the first recurrence equation, focusing particularly on the term $C(5, 3) \cdot H_2$. Split the outcome of a race of 5 horses into two parts: (i) the list of all winners that tie for first place followed by (ii) the list of also-rans who are farther back in the pack. The number of ties in list (i) can be $k = 1, 2, 3, 4,$ or 5. In the case of a three-way tie for first place, there are $C(5, 3)$ ways to choose the names of the horses listed in (i). After these names have been chosen, the remaining list (ii) of also-rans can be any list of 2 horses, with ties allowed. The number of such lists is H_2. Thus, by the fundamental principle of counting, there are $C(5, 3) \cdot H_2$ ways to create a list that starts with a three-way tie for first place. Similar explanations can be given for all the other terms on the right side of the first recurrence equation.

We encourage readers to come up with a solution to the general recursion that could be used for an n-horse race and that might be easily grasped by high school students. A summary of the recurrence relationship giving the number of possible finishes for up to a 10-horse race is given in **figure 6**. Readers who want to pursue more properties of these numbers will want to know that they describe the number of *ordered partitions* of a set of N elements and that these numbers are sometimes called *ordered Bell numbers*.

Do you have your own approach to this problem? We would love to see it. Now that we have begun the school year, we also invite you to work on and share with your students the next problem for this department.

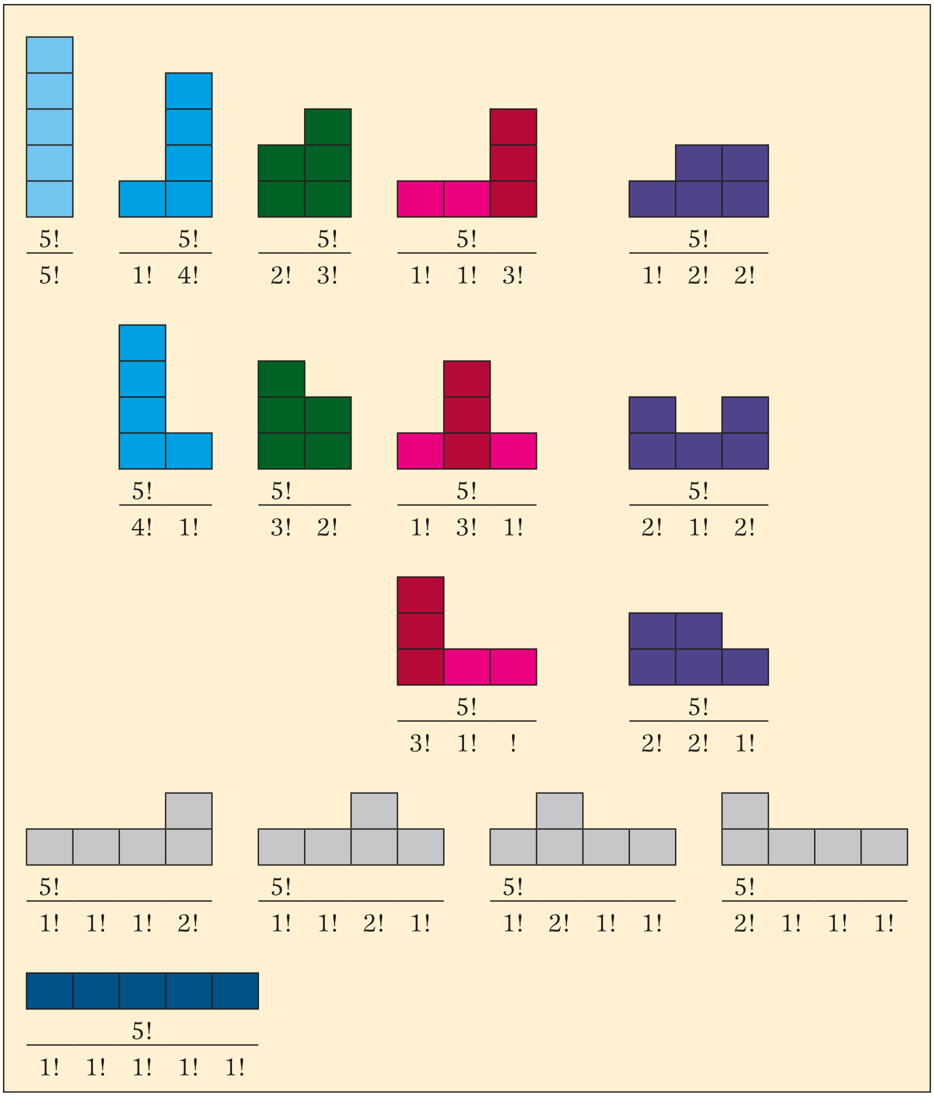

Fig. 5 The diagram to accompany the third solution of the horse race problem puts time on the horizontal axis.

Number of Horses	Number of Possible Finishes (including ties)
0	1
1	1
2	3
3	13
4	75
5	541
6	4,683
7	47,293
8	545,835
9	7,087,261
10	102,247,563

Fig. 6 The Bell numbers listed here produce the number of finishes for an n-horse race for $n \leq 10$.

Solve This Problem!

Work on the following problem with your students. *Reminder:* For your students' work to be considered for publication in *MT* April 2012, please submit all such work to the department editors by December 1, 2011.

Problem: Tipping Points

What's the fastest strategy for spooning out the last melted part of an ice-cream sundae from the bottom of the bowl? Once the liquid level is less than the depth of the spoon, the law of diminishing returns becomes noticeable: As the liquid level drops, the quantity of fluid that the spoon can capture also drops. To offset this frustrating effect, we often resort to the trick of tilting the bowl to pool the fluid into a small deep pocket that fills more of the spoon. It is natural to ask, *What is the tilt angle that will maximize the depth of the pool?* The relationship between the depth of the pool and the tilt angle can be found using geometry and trigonometry.

Assumptions

For simplicity, let's analyze only a two-dimensional version of the problem—we will treat the container as a plane rectangle. Suppose that a quantity of fluid occupies a region of known area A, inside a deep rectangular container of known width W, which is tilted at a small angle T.

Investigation 1

- What is the fluid depth D, measured from the top of the fluid surface to the deepest point that lies directly below the surface? (This deepest point is the lower-left corner in the diagrams below.)
- Express the depth D in terms of the tilt angle, the width W of the container, and the area A occupied by water.
- Note that there are two cases to consider:
  Case 1: The angle is so shallow that the fluid is a trapezoid, as in the left diagram.
  Case 2: The angle is so steep that the fluid is a triangle, as in the right diagram.

Investigation 2

- What is the greatest possible value of D expressed in terms of the width W and area A?
- Does the greatest possible value of D happen in case 1 or case 2? This question can be explored by graphing D as a function of T using the two formulas obtained for D from the results of investigation 1.

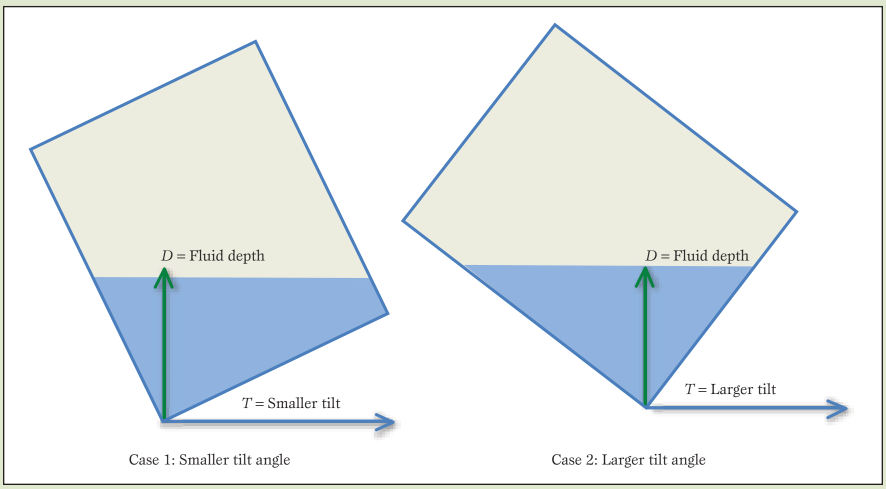

Fig. 1 In both cases, the base of the tilted container is W, and the area of the shaded region is A.