

Probabilities

A la fin de ce chapitre, vous devez être capable de :

- déterminer la probabilité d'événements dans des situations d'équiprobabilité ;
- utiliser des modèles définis à partir de fréquences observées ;
- connaître et exploiter la formule $p(A \cup B) + p(A \cap B) = p(A) + p(B)$.

From frequencies to probabilities

12.1 Consider the following random experiment : roll two 6-sided dice and record the sum of the top sides.

Part A – Statistics

1. Can you predict with certainty the outcome of this experiment ?
2. List all the possible outcomes.
3.
 - a. Do you think that all these outcomes have the same probability to appear ?
 - b. Conjecture the most probable outcome, if there is one.
4.
 - a. Build a **sample of size 10** for this experiment and give the absolute and relative frequencies of each sum in a table with three rows and 112 columns.
 - b. According to the table, what is the most probable outcome ?
 - c. Did everyone in the class get the same frequency distribution ? Is that normal ?
5. To build larger samples, we can use a computer to simulate the experiment. Here are the results for a sample of size 100 000 :

| | | | | | | | | | | | |
|------------|------|------|------|-------|-------|-------|-------|-------|------|------|------|
| Sum | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Abs. Freq. | 2893 | 5447 | 8270 | 11189 | 13809 | 16702 | 13903 | 11195 | 8348 | 5530 | 2714 |

- a. Compute the relative frequency of each sum.
- b. According to the table, what is the most probable outcome ?

Part B – Probabilities

Statistics, which are always studied after carrying out the experiment, *a posteriori*, are not enough to completely understand this experiment, as the results from one sample may be quite different from another sample. In this part, we will study the experiment before actually carrying it out, *a priori*.

1. Give all the possible outcomes of this experiment in a double-entry table.
2. Knowing that each cell in the table has the same probability to appear, give in a second table the probability of each sum. Probabilities may be given as fractions or as decimal numbers to 3DP.
3. Compare the probabilities to the relative frequencies with the simulation of size 100 000.

12.2 In casinos, dice are changed very often to avoid wear, that could induce irregularities. A casino has received from a new manufacturer a batch of dice, all of them truly identical, and an employee is asked to check if these dice are well-balanced. To do so, he picks one die randomly.

1. First, the employee rolls the die 10 times and records the frequency of each side :

| | | | | | | |
|-----------|---|---|---|---|---|---|
| Side | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency | 3 | 1 | 1 | 2 | 1 | 2 |

- a. Compute the relative frequency of each side and give the results in a table. Three more rows will be added to this table in the next questions. The row with the results of the 10 throws will be labelled “10 throws”.
 - b. The employee thinks that the die is well-balanced. What do you think ?
2. The employee goes on by rolling the die 90 extra times, for a total of 100 throws. The results for these 100 throws are given below.

| | | | | | | |
|-----------|----|----|----|----|----|----|
| Side | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency | 15 | 11 | 14 | 15 | 19 | 26 |

- a. Compute the new relative frequencies and show them in a new row in the table you did in question 1.
 - b. The employee now doubts the fact that the die is well-balanced. What assumption could he logically make at that point ?
3. The employee goes on by rolling the die 900 extra times, for a total of 1000 throws. The results for these 1000 throws are given below.

| | | | | | | |
|-----------|-----|----|-----|-----|-----|-----|
| Side | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency | 170 | 79 | 167 | 176 | 238 | 170 |

- a. Compute the new relative frequencies and show them in a new row in the table you did in question 1.
 - b. Is the assumption made by the employee after the 100 throws confirmed by these results ? If not, what can he now surmise ?
4. To get to the bottom of the problem, the employee enlists some of his colleagues and together they roll the die 4000 extra times, to reach a total of 5000 throws. The results are shown below.

| | | | | | | |
|-----------|-----|-----|-----|-----|------|-----|
| Side | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency | 855 | 396 | 818 | 859 | 1223 | 849 |

- a. Compute the new relative frequencies and show them in a new row in the table you did in question 1.
 - b. Do these new values confirm the employee’s inkling ? What should he tell to the casino’s management ?
5. The probabilities of the sides of this die are in fact equal to some simple fractions. Conjecture their values.

Modeling a situation

12.3 A card is randomly picked from a 32-cards deck, with 8 cards in four different suits : hearts (\heartsuit), spades (\spadesuit), diamonds (\diamondsuit) and clubs (\clubsuit).

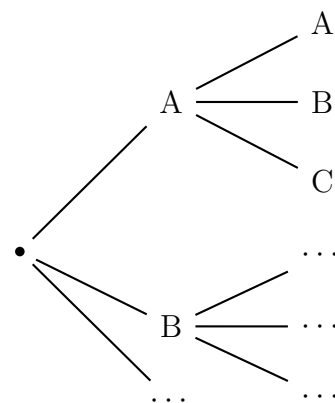
1. What is the probability of picking a club ?
2. What is the probability of picking a black card ?
3. What is the probability of not picking a diamond ?
4. What is the probability of picking a face card (jack, queen or king) ?
5. What is the probability of picking an ace ?
6. What is the probability of not picking a 7 ?
7. What is the probability of picking a black jack ?
8. What is the probability of picking the queen of hearts ?

12.4 In the French National Lottery (Loto), each player picks six numbers (plus one, called the complementary number, that we won't consider in this exercise) between 1 and 49. In this exercise, we consider the random experiment that consists in drawing a single ball from the 49 in the machine.

1. What is the probability of picking an even number ?
2. What is the probability of picking an odd number ?
3. What is the probability of picking a multiple of 5 ?
4. What is the probability of picking a number divisible by 7 ?
5. What is the probability of picking an even multiple of 5 ?
6. What is the probability of picking an odd multiple of 5 ?

12.5 The main door of a building is set up with a keyboard with three letters A , B , C . The code that activates the opening of the door is made of two letters, possibly the same.

1. Copy and fill the tree on the right-hand side, that gives all the possible codes :
2. How many different possible codes are there.
3. Compute the probability of the following events.
 T : The code ends with an A .
 I : The code is made of twice the same letter.
 C : The code starts with an A .



12.6 Inside an urn are four balls numbered 1 to 4. A ball is randomly picked from the urn and then, without putting back the first one, a second ball. The two numbers are multiplied to make the result of the experiment.

1. Use a probability tree to model the situation.
2. What are the possible outcomes in this random experiment ?
3. Compute the probability distribution of this random experiment.

12.7 At a school canteen, each person's menu is made by choosing :

- a starter (coleslaw, pasta salad and spring rolls) ;
- a main course (chicken with fries or roasted pork with peas) ;
- a dessert (yogurt, ice cream or fruit).

1. Build a tree to count the number of possible menus.
2. How many menus include a yogurt ?
3. How many menus include coleslaw and chicken ?
4. How many menus include coleslaw or chicken ?

12.8 Roger is watching an old TV set that receives only 5 programs, 3 of which are showing sport at that moment. He chooses randomly one program to start with and then moves on to a different program. We look at the number of sport programs he will be watching.

1. Use a table or a probability tree to show all the possible situations.
2. Compute the probability that Roger watches
 - a. no sport program ;
 - b. at least one sport program ;
 - c. exactly one sport program ;
 - d. two sport programs.

12.9 Two fair 6-sided dice are rolled and the highest top side is recorded.

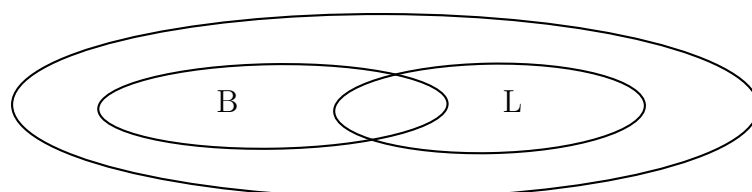
1. Fill out a double-entry table to model the situation.
2. Compute the probability distribution of this random experiment.

12.10 In a summer camp, there are 100 children. They can play two different sports, football or tennis. Of the 100 children, 60 like football, 45 like tennis and 18 like both sports.

1. Represent the situation with a double-entry table
2.
 - a. How many children like none of the two sports ?
 - b. How many children like only football ?
 - c. How many children like only tennis ?
 - d. How many children like at least one of the two sports ?

12.11 During a campaign for road safety, the brakes and lights of 400 cars were monitored. The test showed that 60 cars had a brake defect, 140 cars had a faulty light, and 45 cars had the two defects.

1. Fill out the Venn diagram below to represent the situation.

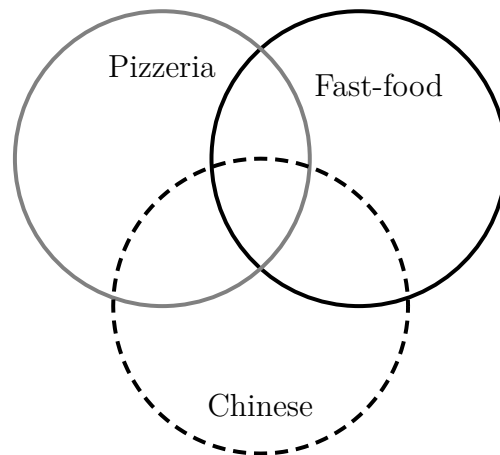


2. One of the 400 cars is randomly picked. Compute the probabilities of the following events.
 - a. The car has a brake defect but no faulty light.
 - b. The car has a faulty light but no brake defect.
 - c. The car has none of these defects.
 - d. The car has at least one of the two defects.

12.12 A group of friends are hanging out during a spring afternoon. It turns out that 60% of them want to go to a movie, 50% want to go to the swimming-pool and 10% want none of these two activities.

1. Illustrate the situation with a Venn diagram.
2. How many friends agree to any of these two activities?
3. How many friends agree only to one of these two activities?

12.13 A group of students are trying to organize an end-of-year party. 56 of them want to go to a pizzeria, 55 to a Chinese restaurant and 50 to a fast-food. Furthermore, 19 refuse to go to a fast-food but agree with the two other possibilities and 12 accept the three possibilities. Finally, 15 refuse both the pizzeria and the fast-food and 7 refuse to go to a Chinese restaurant.



1. Fill out the Venn diagram below.
2. What is the probability that a randomly picked student only agrees to go to a fast-food?
3. If you were the organizer of that party, what restaurant would you choose?

12.14 In a dancing school, 48 students are enrolled in three different classes : break dance (25 students), jazz (18 students) and classical (22 students). Three students attend the three classes, five only do jazz and four are enlisted in classical and jazz but not break dance.

1. Build a Venn diagram to represent the situation.
2. What is the probability that a randomly picked student is doing classical and break dance but not jazz.

12.15 A software is used to sort messages on mail boxes. The designers have tested 10000 e-mails and here are their conclusions :

- 70% of incoming messages are spam ;
- 95% of spam are sent to the Junk folder ;
- 2% of good messages are sent to the Junk folder.

1. Fill out the following table :

| | Spam | Not spam | Total |
|----------|------|----------|-------|
| Junk | | | |
| Not junk | | | |
| Total | | | 10000 |

2. We consider that this sample of 10000 is representative of all of the incoming messages, so we will use the relative frequencies as probabilities.
 - a. Compute to 3DP the approximate values of the probabilities of the events :
 - i. a spam message is not sent to the Junk folder ;
 - ii. a good message is sent to the Junk folder ;
 - iii. the software is making a mistake.
 - b. Compute the probability that a message sent to the Junk folder was spam.
 - c. Compute the probability that a message not sent to the Junk folder was good.

12.16 A fair 4-sided die, with sides numbered 1 to 4, is thrown three times to make a three-digits number.

1. Build a probability tree to represent this situation.
2. List all the possible results.
3. Compute the probabilities of the following events.

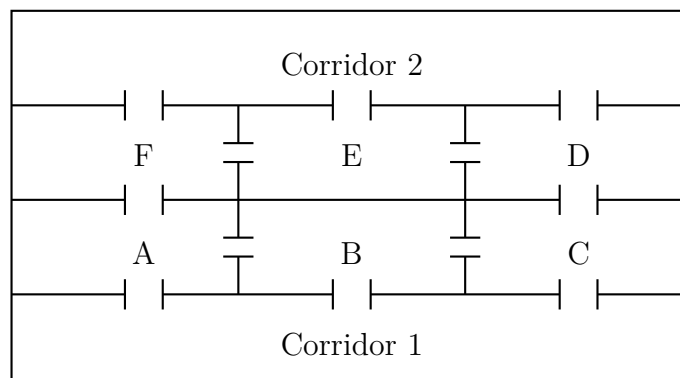
A : The number starts with a 1.

B : The number is a multiple of 3.

C : The number is even.

D : The number is strictly greater than 320.
4. A casino is offering this game to its clients, with the following rules :
 - the player earns 10 euros if the number is made of three times the same digit ;
 - the player earns 5 euros if the same digit appears exactly twice in the number ;
 - for all other numbers, the player earns 1 euro.
 - a. Compute the probability of each earning.
 - b. Compute the average earning of the player.
 - c. The casino decides to set the price for a game to 3 euros. Will it win or lose money ?

12.17 An office is made of two corridors, numbered 1 and 2, and six working rooms numbered A, B, C, D, E and F. These rooms communicate through 12 doors, as shown in the picture below :



There is no door between rooms B and E.

- A robot, stored in corridor 1, is programmed to clean exactly three working rooms.
- The robot starts its course by entering one of the rooms A, B or C.
- As soon as the robot enters a room, it cleans it completely.
- The robot cannot go through the same door twice, or clean the same room twice.
- Once the three rooms have been cleaned, the robot returns to the corridor 1, sometimes passing through corridor 2 and a side way not shown on the picture.

A *circuit* is any possible list of three rooms cleaned successively by the robot.

Examples :

- ABC and BCD are circuits.
- CBA and ABC are different circuits.
- ABE is not a possible circuit (there is no door between rooms B and E).
- DEF is not a possible circuit (the robot can't start by cleaning room D).

1. Find the six possible circuits (you may use a tree to do so).

We admit that these six circuits have the same probability.

2. a. Compute the probability p_1 of the event “room E is the third room cleaned by the robot”.

b. Compute the probability p_2 of the event “the robot exits through corridor 2”.

3. The table below gives the time needed to clean each of the rooms (in minutes) :

| Room | A | B | C | D | E | F |
|------|--------|--------|--------|--------|--------|--------|
| Time | 20 min | 24 min | 30 min | 14 min | 22 min | 14 min |

a. Compute the time needed for each of the six possible circuits.

b. Compute the average time for a circuit.

c. Compute the probability p_3 of the event “the robot cleans the 3 rooms in less than 60 minutes”.

Probabilities calculations

12.18 A survey has been made in early 1998 about the Internet. The answers of 1000 people are given below :

- 60% have reported not using the Internet ;
- 35% are under 25 years old and among them 80% have reported using the Internet ;
- 30% are over 50 years old and among them 85% have reported not using the Internet.

1. Fill out the following table :

| | Internet | No Internet | Total |
|--------------------|----------|-------------|-------|
| Under 25 years old | | | |
| 25 to 50 years old | | | |
| Over 50 years old | | | |
| Total | | | 1 000 |

2. We pick randomly a person from the survey and we define the events :
- A : “The person is an Internet user.”
 B : “The person is under 25 years old.”
- Compute the probabilities of the events A and B .
 - Describe with a sentence the event \overline{B} , then compute $p(\overline{B})$.
 - Describe with a sentence the event $A \cap B$, then compute $p(A \cap B)$.
 - Describe with a sentence the event $A \cup B$, then compute $p(A \cup B)$.
 - Knowing that the person chosen is not an Internet user, what is the probability that he is under 50 years old?

12.19 Four tokens, marked A, B, C, D, are put in a bag. Three tokens are picked successively, without putting them back in the bag.

- Use a tree to describe all the possible outcomes.
- We consider the following events :
 - E : “The first token is marked with the letter B.”
 - F : “The token C hasn’t been picked.”
 Compute the probabilities of the events E and F .
- What outcomes make up the event $E \cap F$?
 - Compute the probability of the event $E \cap F$.
- Deduce the probability of the event $E \cup F$.

12.20 A random experiment is defined as follows : throw a fair six-sided die, then a coin and count 1 for heads and 2 for tails. Multiply the number from the die by the number from the coin.

- Represent the experiment with a probability tree.
- At the end of each branch of the tree, show the outcome of the experiment.
- In a table, show the probability distribution of the experiment.
- Compute the probabilities of the following events :

A : the result is even ;
 B : the result is a multiple of 3 ;
 C : the result is neither 6, nor 5 ;
 D : the result is greater than or equal to 4 ;
 E : the result is strictly less than 6.
- List all the outcomes making up the event $A \cap B$.
 - Deduce the probability of the event $A \cap B$.
- List all the outcomes making up the event $A \cup B$.
 - Deduce the probability of the event $A \cup B$.
 - Is it true that $P(A \cup B) = P(A) + P(B)$? If so, prove it. If not, correct the formula.

12.21 A six-sided fair die is rolled. Let's consider the following events :

- A : "You get the number 1."
- B : "You get a number strictly greater than 2."
- C : "You get an even number."

1. List all the outcomes making up the events A , B and C .
2. Represent the set of all possible outcomes E and the sets A , B and C with a diagram.
3. List all the outcomes making up the events $A \cup B$ and $A \cap B$.
4. Find the probabilities of the events A , B , C , $A \cup B$ and $A \cap B$.

12.22 An industrial production line consists of two machines called M1 and M2 in this exercise. Surveys have shown that

- the probability of the event A "the machine M1 is operational" is 96,7 %;
 - the probability of the event B "the machine M2 is operational" is 98,2 %.
1. The probability that the two machines are operational at the same time is 95,5 %. Translate this sentence into mathematical notation.
 2. Translate the following events into English, then find their probabilities.

$$A, B, \bar{A}, \bar{B}, A \cup B, \overline{A \cup B}.$$

12.23 A car salesman has noticed that 70% of his clients buy a brand new car, of which 80% choose a French brand. All in all, the French brands account for three quarters of his sales. Let's consider a random client and define the following events :

- N : "the client buys a brand new car";
- F : "the client buys a French car".

1. Represent the information in the text with a table or a diagram.
2. Translate each of the following events into English and compute their probabilities.

$$N, F, \bar{N}, \bar{F}, N \cap F, \overline{N \cap F}, N \cup F, \overline{N \cup F}.$$

12.24 A company produces pocket calculators in large quantity. The calculators can have two types of flaws. : the A flaw or the B flaw. After a statistical study on 800 calculators it turns out that :

- 15% of the calculators have the A flaw ;
- among the calculators having the A flaw, 95 % do not have the B flaw ;
- among the calculators that do not have the A flaw, 5 % have the B flaw.

1. Copy and fill out the following double-entry table :

| | | | |
|----------------|--------|----------------|-------|
| | A flaw | Not the A flaw | Total |
| B flaw | | | |
| Not the B flaw | | | |
| Total | | | 800 |

2. One calculator is randomly picked.

- a. Give the probability of the event A : “The calculator has the A flaw”.
 - b. Give the probability of the event B : “The calculator has the B flaw”.
 - c. Define the events $A \cap B$ and $A \cup B$, then compute their probabilities.
 - d. What is the probability of the event : “The calculator has no flaw”.
3. A calculator with the A flaw has been chosen. What is the probability that it also has the B flaw?

12.25 In a class, there are 24 students, including 9 boys. Among the boys, 7 are half-boarders, and none are boarders. In the whole class, there are twice less day students than boarders and half-boarders all together, and there is only one girl which is a full-boarder. A student from that class is randomly picked, and we define the following events :

G : the student is a girl ; **H** : the student is a half-boarder ;
B : the student is a boy ; **F** : the student is a boarder.
D : the student is a day student ;

1. Sum up the information from the text in a double-entry table.
2. Translate into English the following events

- | | | | |
|------------------------|------------------------|----------------------------|----------------------------|
| a. $F \cup H$; | c. $G \cap D$; | e. $G \cap B$; | g. \overline{F} . |
| b. $H \cup D$; | d. $B \cap H$; | f. \overline{G} ; | |

3. Use the table to compute the probabilities of these events.

12.26 A coin is thrown repeatedly, a certain number of times.

1. If the coin is thrown just once, what is the probability of getting a head?
2. If the coin is thrown 2 times, what is the probability of getting only heads?
3. If the coin is thrown 3 times, what is the probability of getting only heads?
4. If the coin is thrown n times, with n a natural number, what is the probability of getting only heads?
5. How many times must the coin be thrown so that the probability of getting only heads is less than 0.001?
6. Consider the algorithm below :

```

begin
    Input:  $e$ , a real number ;
     $1 \rightarrow n$  ;
     $0.5 \rightarrow p$  ;
    while  $p \geq e$  do
        |  $n + 1 \rightarrow n$  ;
        |  $0.5^n \rightarrow p$  ;
    end while
    Output:  $n$  ;
end
```

- a. Apply the algorithm with $e = 0.1$, with $e = 0.01$ and with $e = 0.001$.
- b. What is the aim of this algorithm?
- c. How would you modify the algorithm for a biased coin, where the probability of getting a head is entered by the user?

Homework #8 – The minimal depth

Roger, the farmer, noticed that his cows drink at least 2000 liters per day. Below is the shape of the trough that he want to build. The bottom $DD'CC'$ is a 1×2 m. rectangle and the front face $ABCD$ is a isosceles trapezium. What will be the minimal depth (denoted h on the plan) of the trough?

